33-467: Astrophysics of Stars and the Galaxy
Due: Wednesday 26th October.

Problem Set 6

1. As we have shown in class, the Rosseland mean opacity, κ_R, is defined to be,:

$$\kappa_R \equiv \frac{\int_0^\infty d\nu \frac{\partial B_\nu}{\partial T}}{\int_0^\infty \frac{d\nu}{\kappa_\nu} \frac{\partial B_\nu}{\partial T}}, \tag{1}$$

where κ_ν is the frequency-dependent opacity and B_ν is the Planck function,

$$B_\nu = \frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1}. \tag{2}$$

The aim of this problem is to calculate the Rosseland mean opacity for the case of free-free absorption, in which a photon is absorbed by a free electron in the Coulomb field of a nucleus. The frequency dependent free-free absorption co-efficient for pure hydrogen is,

$$\kappa_\nu = 1.32 \times 10^{56} \frac{\rho g_{ff}}{\nu^3 T^{1/2}}(1 - e^{-h\nu/kT}) \;\; cm^{-1} \tag{3}$$

In this expression g_{ff} is a quantum mechanical correction called the *Gaunt factor*, which you may assume to be a constant for the purpose of this problem.

(a) Derive an expression for $\partial B_\nu / \partial T$.

(b) Introduce a new dimensionless variable $x \equiv h\nu/kT$. Write the expression

$$\frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T} \tag{4}$$

for free-free emission in terms of x and plot the resulting function (please include this plot when you hand in your homework). Use this plot to argue that the Rosseland mean is determined largely by κ_ν where ν is a few times kT/h.

(c) Show that the Rosseland mean opacity obeys *Kramer's law*

$$\kappa_R \propto \rho T^{-3.5}. \tag{5}$$

2. The material in the envelope of a star obeying hydrostatic equilibrium has a ratio of specific heats $\gamma = 4/3$ and satisfies the equation of state for an ideal gas. The star is sufficiently centrally condensed that the mass in the envelope is negligible compared to the core mass, M_c. The envelope is convectively unstable in such a way as to satisfy the criterion for convective instability. Show that the temperature within the envelope varies with radius, r, as

$$T = \frac{\mu m_H G M_c}{4k}\left(\frac{1}{r} - \frac{1}{r_S}\right) + T_S \tag{6}$$

where μm_H is the mean molecular weight of the stellar material, k, is the Boltzmann constant, G is the gravitational constant, r_S is the surface radius and T_S is the surface temperature of the star.

3. Show that, for a star with a radiative envelope in which the opacity $\kappa = \kappa_0 P/T^4$, the variation of pressure P and temperature T obeys the approximate relation

$$P = \left(\frac{4\pi acGM}{3L\kappa_0}\right)^{1/2} T^4 \tag{7}$$

where a is the radiation constant, c is the velocity of light, G is the gravitational constant, M is the mass of the star and L is the luminosity of the star.