33-467: Astrophysics of Stars and the Galaxy
Due: Wednesday 12th October.

Problem Set 5

1. Recall that in polytropic stars, pressure and density are simply related as $P = K\rho^{1+1/n}$, where K and n are constants and n is the polytropic index. These stars satisfy the Lane-Emden equation (see notes from class) where the dimensionless variables θ and ξ are defined in tersm of radial coordinate r and density ρ as $r = \alpha\xi$ and $\rho = \lambda\theta^n(r)$ where $\lambda = \rho_c$ is the central density. The relevant boundary conditions are $\theta(0) = 1$, $\theta'(0) = 0$, at the center ($\xi = 0$) and $\theta(\xi_1) = 0$ at the surface (the first zero crossing of θ, which occurs at $\xi = \xi_1$).

   Show the following:

   (a) The radius of the star is $R = [(n+1)K/(4\pi G)]^{1/2}\rho_c^{[(1-n)/n]}\xi_1$
   (b) The total mass of a polytropic star is $M = -4\pi\alpha^3\rho_c\xi_1^2(d\theta/d\xi)_{\xi_1}$.
   (c) The ratio of the mean density to the central density is $\frac{\bar{\rho}}{\rho_c} = -(3/\xi_1)(d\theta/d\xi)_{\xi_1}$.
   (d) The central pressure is $P_c = [4\pi(n+1)(d\theta/d\xi)^2_{\xi_1}]^{-1}(GM^2/R^4)$.
   (Extra credit:)
   (e) For an ideal gas equation of state, the variable θ is a dimensionless temperature, such that $T(r) = T_c\theta(r)$.

2. A star supported by a mixture of gas and radiation pressure is an $n = 3$ polytrope where K is a function of the constant $\beta = P_{gas}/P$ (where $P = P_{gas} + P_{rad}$ i.e., the Eddington-standard model). Using the equations derived above and the Table posted on the class webpage, calculate β for a star with $M = 30M_\odot$. What is the dominat pressure in such a massive star?

3. Use the linear relation for the mass density in Eq. (2) and the pressure as a function of radius $P(r)$ derived in Question 3 of Problem Set 1 (derived by applying hydrostatic equilibrium) show that, if the equation of state is that of a perfect gas, the temperature is given by:

$$T(r) = \frac{\mu m_H G M}{12kR}(5 + 5x - 19x^2 + 9x^3), \tag{1}$$

   where $x = r/R$, M is the total mass of the star and μ is the mean molecular weight of the gas. Use the temperature gradient predicted at $r = 1/2R$ by the relation above, along with the equation of radiative equilibrium, to obtain a relation between mass, M and luminosity L. You may assume that $L(r) = L$ =constant, and that pure electron scattering dominates the opacity (i.e. $\kappa = \kappa_0 =$ constant)